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Scattering of Long Waves in a Rotating Bifurcated Channel

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An exact solution is obtained for the problem of the scattering of a Kelvin wave by a rotating bifurcated channel and it is shown that an appropriate secondary Kelvin wave is created in all the parts of the field.

1. INTRODUCTION

The problem being discussed in this paper concerns the transmission and reflection of long waves (Lamb, 1932, Section 172) in a duct consisting of two infinite vertical plates in the presence of a semi-infinite barrier of zero thickness (bifurcated channel).

The channel is assumed to be rotating at constant angular velocity (as a channel fixed to the earth) and the depth of the fluid is uniform.

The disturbance is due to a Kelvin-wave traveling in the lower part of the duct in Figure 1.

Regarding problems of this type various techniques have been used as the integral equation methods or Green's function methods (Crease, 1958; Buchwald, 1971). The present problem is attacked by application of the complex Fourier transform directly to the equations which, according to the linearized theory of long waves in a rotational system, govern the effect examined (Jones, 1952), and a suitable functional equation is derived. An exact solution, given for all parts of the field, shows the propagation of Kelvin waves in the whole field even in the "shadow" zone. This is due in a measure to the fact that even if there is only a single semi-infinite barrier in a rotating system, Kelvin waves are propagated in the shadow zone (Kapoulitsas, 1977).

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The results obtained may be useful to interpret qualitatively the formation of tides in the case of an oblong island between two straight coastlines.

2. EQUATIONS OF MOTION AND STATEMENT OF THE PROBLEM

Assuming a time factor of $e^{-i\omega t}$, the linearized equations governing the propagation of long waves in a system rotating with an angular velocity $2f$ are, in rectangular Cartesian coordinates.

$$\frac{\partial^2\phi_t}{\partial x^2}+\frac{\partial^2\phi_t}{\partial y^2}+k^2\phi_t=0$$

$$hk^2u=-i\omega\frac{\partial\phi_t}{\partial x}+f\frac{\partial\phi_t}{\partial y} \tag{2.1}$$

$$hk^2v=-f\frac{\partial\phi_t}{\partial x}-i\omega\frac{\partial\phi_t}{\partial y}$$

In these equations ϕ_t is the total surface elevation above its mean level, $u(x, y)$, $v(x, y)$ are the particle velocities in x and y directions, and h is the depth of the water assumed constant.

Also $c^2k^2=\omega^2-f^2, c^2=gh$, where g is the gravitational acceleration, and $\omega>f$.

Let us assume that a Kelvin wave

$$\phi_i=\exp\left[\frac{1}{c}(i\omega x-fy)\right]$$

travels towards the origin from inside of the region

$$0\leqslant y\leqslant b, \qquad x\leqslant 0$$

of a bifurcated channel consisting of the planes

$$y=0, \quad -\infty<x<\infty \quad \text{and} \quad y=a, \quad -\infty<x<\infty$$

and the half-plane

$$y=b, \qquad x<0$$

where $a>b>0$ (Figure 1).

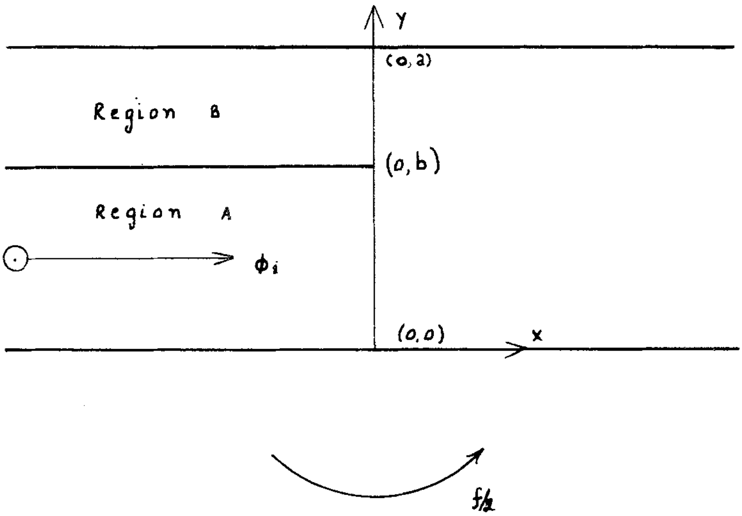

Fig. 1. The rotating bifurcated channel.

The whole field is composed of two regions: The region A $(0 \leqslant y \leqslant b, -\infty < x < \infty)$ and the region B $(b \leqslant y \leqslant a, -\infty < x < \infty)$.

The motion of the fluid, occupying the entire field, is assumed to satisfy equations (2.1) and we define the function ϕ as follows:

$$\phi_t = \begin{cases} \phi + \phi_i & \text{in region A} \\ \phi & \text{in region B} \end{cases} \tag{2.2}$$

For convenience we put $f = kc \sinh \beta$, $\omega = kc \cosh \beta$; the incident wave is then

$$\phi_i = \exp[k(ix \cosh \beta - y \sinh \beta)]$$

The problem is now to determine the solution of the first of equations (2.1) satisfying the following conditions:

$$\frac{\partial \phi}{\partial y} - i \tanh \beta \frac{\partial \phi}{\partial x} = 0 \begin{cases} \text{on } y = 0, & -\infty < x < \infty \\ \text{on } y = a, & -\infty < x < \infty \\ \text{on } y = b \pm 0, & x < 0 \end{cases} \tag{2.3}$$

which express that the normal component (v) of the velocity of the fluid vanishes on the boundaries. In addition

$$\left(\frac{\partial\phi}{\partial y}-i\tanh\beta\frac{\partial\phi}{\partial x}\right)_{y=b-0}=\left(\frac{\partial\phi}{\partial y}-i\tanh\beta\frac{\partial\phi}{\partial x}\right)_{y=b+0}, \qquad -\infty<x<\infty \tag{2.4}$$

expressing the continuity of the normal velocity across the line $y=b$, $-\infty<x<\infty$, and

$$\phi(x,b-0)+\phi_i(x,b)=\phi(x,b+0), \qquad x>0 \tag{2.5}$$

which denotes the continuity of the field function $\phi_t(x,y)$ across the half-plane $y=b$, $x>0$.

Finally the solution $\phi(x,y)$ must satisfy the following edge conditions at the edge $(0,b)$:

$$\left.\begin{aligned}&\phi=0(1) && \text{as } x\to 0\pm 0 \text{ on } y=b\\ &\frac{\partial\phi}{\partial y}=0(x^{-1/2}) && \text{as } x\to 0\pm 0 \text{ on } y=b\end{aligned}\right\} \tag{2.6}$$

In the following we assume for mathematical convenience that ω is complex with a small positive imaginary part ω_2 (i.e., $\omega=\omega_1+\omega_2$, $\omega_1\gg\omega_2>0$) and this implies that k must also be complex with a small positive imaginary part (i.e., $k=k_1+ik_2$, $k_1\gg k_2>0$).

The solution is obtained from the final results by taking $\omega_2\to 0+0$, which implies $k_2\to 0+0$.

From (2.2) and the well-known result that all possible waves traveling along a channel are of order $\exp(-\tau_0|x|)$ as $|x|\to\infty$ in the direction of their propagation, where

$$\tau_0<k_2(1-f^2/\omega^2)^{1/2}<k_2$$

it follows that

$$\phi=0(e^{-\tau_0|x|}) \qquad \text{as } x\to\pm\infty \ (0\leqslant y\leqslant a)$$

Thus the two-sided complex Fourier transform $\Phi(\alpha,y)$ of $\phi(x,y)$ in x, defined by

$$\Phi(\alpha,y)=\int_{-\infty}^{\infty}\phi(x,y)e^{i\alpha x}dx, \qquad \alpha=\sigma+i\tau \quad (\sigma,\tau \text{ real}) \tag{2.7}$$

as well as the correspondent one-sided Fourier transforms $\Phi_-(\alpha, y)$ and $\Phi_+(\alpha, y)$ of $\phi(x, y)$ in x, from $-\infty$ to 0 and from 0 to $+\infty$, respectively, exist and are regular in the regions $|\tau|<\tau_0$, $\tau<\tau_0$, and $\tau>-\tau_0$ of the α plane, respectively. Evidently

$$\Phi_-(\alpha, y)+\Phi_+(\alpha, y)=\Phi(\alpha, y) \tag{2.8}$$

3. THE BASIC FUNCTIONAL EQUATION

Taking the two-sided Fourier transform to the first of Equations (2.1) we get

$$\frac{d^2\Phi(\alpha, y)}{dy^2}-\gamma^2\Phi(\alpha, y)=0 \tag{3.1}$$

with

$$\gamma^2=(\alpha^2-k^2)^{1/2} \tag{3.2}$$

As we shall see later $\Phi(\alpha, y)$ is even in γ, and thus the branch points of γ at $\alpha=\pm k$ do not play any role.

Equation (3.1) holds in the whole field and its general solution is

$$\Phi(\alpha, y)=\begin{cases} A(\alpha)e^{-\gamma y}+B(\alpha)e^{\gamma y} & (0\leqslant y\leqslant b, \text{region A}) \\ C(\alpha)e^{-\gamma y}+D(\alpha)e^{\gamma y} & (b\leqslant y\leqslant a, \text{region B}) \end{cases} \tag{3.3}$$

where the unknown A, B, C, D are functions only of α.

The two-sided Fourier transforms of the first two of the boundary conditions (2.3) are, respectively,

$$\Phi'(\alpha,0)-\alpha(\tanh\beta)\Phi(\alpha,0)=0$$

$$\Phi'(\alpha, a)-\alpha(\tanh\beta)\Phi(\alpha, a)=0 \tag{3.4}$$

The primes of Φ's are taken to mean the derivative with respect to y. Putting now

$$\lambda\equiv\frac{\gamma+\alpha\tanh\beta}{\gamma-\alpha\tanh\beta} \tag{3.5}$$

and using equation (3.3) we get

$$B=\lambda A \tag{3.6}$$

and

$$D = \lambda C e^{-2\gamma a} \tag{3.7}$$

Next applying the two-sided Fourier transform to equation (2.4) we obtain

$$\Phi'(\alpha, b-0) - \tanh\beta \cdot \Phi(\alpha, b-0) = \Phi'(\alpha, b+0) - \alpha \tanh\beta \cdot \Phi(\alpha, b+0) \tag{3.8}$$

Using then equation (3.3) we have

$$B - C = (A - C)\lambda e^{-2\gamma b} \tag{3.9}$$

From (3.6), (3.7), (3.8), and (3.9) we also express C and D in terms of A, i.e.,

$$C = -A \frac{\sinh \gamma b}{\sinh \gamma(a-b)} e^{\gamma a} \tag{3.10}$$

$$D = -\lambda A \frac{\sinh \gamma b}{\sinh \gamma(a-b)} e^{-\gamma a} \tag{3.11}$$

Next, on taking the one-sided Fourier transform (from $-\infty$ to 0) of the third of equations (2.3) we have

$$\Phi'_{-}(\alpha, b \pm 0) - \alpha \tanh\beta \cdot \Phi_{-}(\alpha, b \pm 0) - i \tanh\beta \cdot \phi(0, b) = 0 \tag{3.12}$$

since $\phi(0, b+0) = \phi(0, b-0) = \phi(0, b)$.

In equation (3.12) the upper (lower) signs go together.

From equations (3.8) and (3.12) we obtain

$$\begin{aligned} &\Phi'_{+}(\alpha, b-0) - \alpha(\tanh\beta)\Phi_{+}(\alpha, b-0) + i(\tanh\beta)\phi(0, b) \\ &\quad = \Phi_1(\alpha, b+0) - \alpha(\tanh\beta)\Phi_{+}(\alpha, b+0) + i(\tanh\beta)\phi(0, b) \equiv P_{+}(\alpha), \end{aligned} \tag{3.13}$$

say.

Lastly the one-sided Fourier transform (from 0 to ∞) of equation (2.5) gives us

$$\Phi_{+}(\alpha, b-0) + \frac{i \exp(-kb \sinh\beta)}{k \cosh\beta + \alpha} = \Phi_{+}(\alpha, b+0) \tag{3.14}$$

Then by virtue of (2.8), (3.3), (3.6), (3.12), and (3.13) we obtain finally

$$P_+(\alpha)=2A(\gamma+\alpha\tanh\beta)\sinh\gamma b \tag{3.15}$$

Now we define the function $Q_-(\alpha)$ as

$$Q_-(\alpha)=\tfrac{1}{2}\{\Phi_-(\alpha,b-0)-\Phi_-(\alpha,b+0)\} \tag{3.16}$$

Then by virtue of equations (3.3), (3.5), (3.6), (3.7), (3.8), and (3.14) we get finally

$$2Q_-(\alpha)-\frac{i\exp(-kb\sinh\beta)}{k\cosh\beta+\alpha}=\frac{2A}{\gamma-\alpha\tanh\beta}\cdot\frac{\gamma\sinh\gamma a}{\sinh\gamma(a-b)} \tag{3.17}$$

Eliminating A between (3.15) and (3.17) we obtain

$$2Q_-(\alpha)-\frac{i\exp(-kb\sinh\beta)}{k\cosh\beta+\alpha}=\frac{P_+(\alpha)}{\gamma^2-\alpha^2\tanh\beta}\cdot\frac{\gamma\sinh\gamma a}{\sinh\gamma b\cdot\sinh\gamma(a-b)} \tag{3.18}$$

Equation (3.18) is a functional equation of the Wiener–Hopf type.

4. THE SOLUTION IN THE FOURIER TRANSFORM DOMAIN

We proceed now to solve equation (3.18). First of all we have to factorize the term

$$M(\alpha)=\frac{\sinh\gamma b\sinh\gamma(a-b)}{\gamma\sinh\gamma a} \tag{4.1}$$

so that

$$M(\alpha)=M_+(\alpha)\cdot M_-(\alpha) \tag{4.2}$$

where $M_+(\alpha)$ and $M_-(\alpha)$ are regular and free of zeros in an upper and a lower half-plane, respectively, the half-planes having a common strip.

We note that $M(\alpha)$ is even in γ and thus it has no branch points. More precisely $M(\alpha)$ is a meromorphic function of α and so it can be factorized by applying the infinite product expansion of an integral function with infinitely many zeros (Titchmarsh, 1968; Noble, 1958).

The factorization of $M(\alpha)$ is known (cf. Mittra and Lee, 1971) and the expressions for $M_+(\alpha)$ and $M_-(\alpha)$ are as follows:

$$M_+(\alpha)$$

$$=M_-(-\alpha)=\left[\frac{\sin kb\cdot\sin k(a-b)}{k\sin ka}\right]^{1/2}\exp\left\{i\frac{\alpha}{\pi}\left[b\log\frac{a}{b}+(a-b)\log\frac{a}{a-b}\right]\right\}$$

$$\times\frac{\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_{nb}}\right)e^{(ib\alpha/n\pi)}\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_{n(a-b)}}\right)e^{i(a-b)\alpha/n\pi}}{\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_{na}}\right)^{ia\alpha/n\pi}} \tag{4.3}$$

where $\alpha_{nl}=i(\pi^2n^2/l^2-k^2)^{1/2}$ and $(\pi^2n^2/l^2-k^2)^{1/2}$ is taken to have positive real part or negative imaginary part and $l=a$ or b or $(a-b)$. Also

$$M_+(\alpha)=M_-(-\alpha)=0(\alpha^{-1/2}) \qquad \text{as}\,|a|\to\infty\,\text{in}\,\tau>-k_2 \tag{4.3a}$$

From equation (4.3) it is clear that $M\pm(\alpha)$ have no branch points. $M_+(\alpha)$ has poles at $\alpha=-\alpha_{na}$ and zeros at $\alpha=-\alpha_{nb}$ and $\alpha=-\alpha_{n(a-b)}$ $(n=1,2,\ldots)$. But all these points lie in the lower half α plane $(\tau<-k_2)$ and so $M_+(\alpha)$ is regular and nonzero in the upper half α plane $(\tau>-k_2)$.

$M_-(\alpha)$, therefore, has a similar behavior in the lower half α plane $\tau<k_2$.

Now equation (3.18) may be rewritten as

$$2(\alpha-k\cosh\beta)M_-(\alpha)Q_-(\alpha)-\frac{ie^{-kb\sinh\beta}(\alpha-k\cosh\beta)M_-(\alpha)}{k\cosh\beta+\alpha}$$

$$=\frac{\cosh^2\beta}{(\alpha+k\cosh\beta)M_+(\alpha)}P_+(\alpha) \tag{4.4}$$

Next we decompose the term $\dfrac{(\alpha-k\cosh\beta)M_-(\alpha)}{k\cosh\beta+\alpha}$ into the sum

$$\frac{(\alpha-k\cosh\beta)M_-(\alpha)+2k\cosh\beta\cdot M_-(-k\cosh\beta)}{\alpha+k\cosh\beta}$$

$$-\frac{2k\cosh\beta\cdot M_-(-k\cosh\beta)}{\alpha+k\cosh\beta}$$

The first fraction of the above expression is regular in $\tau<k_2$; the second one is regular in $\tau>-\tau_0$ [since $\mathrm{Im}(-k\cosh\beta)=-\omega_2/c<-k_2(1-f^2/\omega^2)^{1/2}\equiv-\tau_0$]. Equation (4.4) is then rewritten as

$$
\begin{aligned}
&2(\alpha-k\cosh\beta)M_-(\alpha)Q_-(\alpha)-i\exp(-kb\sinh\beta)\\
&\times\left\{\frac{(\alpha-k\cosh\beta)M_-(\alpha)+2k\cosh\beta M_-(-k\cosh\beta)}{\alpha+k\cosh\beta}\right\}\\
&=\frac{\cosh^2\beta}{(\alpha+k\cosh\beta)M_+(\alpha)}P_+(\alpha)-i\exp(-kb\sinh)\frac{2k\cosh\beta\cdot M_-(-k\cosh\beta)}{\alpha+k\cosh\beta}
\end{aligned}
\tag{4.5}
$$

The left-hand side of equation (4.5) is regular in $\tau>-\tau_0$, while its right-hand side is regular in $\tau<\tau_0$, and hence both sides are regular in the strip $|\tau|<\tau_0$. It follows from analytic continuation that equation (4.5) is defined in the entire α plane and both sides are equal to an integral function $p(\alpha)$ say. Let us consider now the asymptotic behavior of the functions in equation (4.5). By virtue of equation (4.3) and using the edge conditions (2.6), we find that, according to Lionville's theorem on polynomials, $p(\alpha)$ is zero, and then from (4.5) we obtain

$$P_+(\alpha)=\frac{2ikM_+(k\cosh\beta)\exp(-kb\sinh\beta)}{\cosh\beta}M_+(\alpha) \tag{4.6}$$

Now on the basis of the known function $P_+(\alpha)$ we determine the unknown functions A, B, C, and D from equations (3.15), (3.6), (3.10), and (3.11), respectively. Thus if we put

$$E=\frac{ikM_+(k\cosh\beta)\exp(-kb\sinh\beta)}{\cosh\beta} \tag{4.7}$$

we have

$$A=E\frac{M_+(\alpha)}{(\gamma+\alpha\tanh\beta)\sin\gamma b},\qquad B=E\frac{M_+(\alpha)}{(\gamma-\alpha\tanh\beta)\sin\gamma b} \tag{4.8}$$

$$C=-E\frac{M_+(\alpha)e^{\gamma a}}{(\gamma+\alpha\tanh\beta)\sinh\gamma(a-b)},\qquad D=-E\frac{M_+(\alpha)e^{-\gamma a}}{(\gamma-a\tanh\beta)\sin\gamma(a-b)} \tag{4.9}$$

5. DETERMINATION OF THE SOLUTION $\phi(x, y)$ OF THE FIELD

To obtain the field function $\phi(x, y)$ from its Fourier transform let us apply the inverse Fourier transform

$$\phi(x, y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi(\alpha, y)e^{-i\alpha x}d\alpha \tag{5.1}$$

where the path of integration is chosen to be the real axis $\tau=0$.

In the following we determine $\phi(x, y)$ in the various parts of the field.

5.1. Region A ($0\leqslant y\leqslant b, -\infty<x<\infty$). From equations (3.3) and (4.8) we get

$$\Phi(\alpha, y)=E\frac{M_+(\alpha)}{\sin\gamma b}\left(\frac{e^{-\gamma y}}{\gamma+\alpha\tanh\beta}+\frac{e^{\gamma y}}{\gamma-\alpha\tanh\beta}\right) \tag{5.2}$$

It is evident that $\Phi(\alpha, y)$ is even in γ and so has no branch points. The only singularities in it are (simple) poles coming from the poles of $M_+(\alpha)$, the zeros of sin γb (except at $\gamma=0$), and the zeros of ($\gamma\pm\alpha$ tanh β). For $x<0$ we close the contour in the upper half-plane (Figure 2) where the

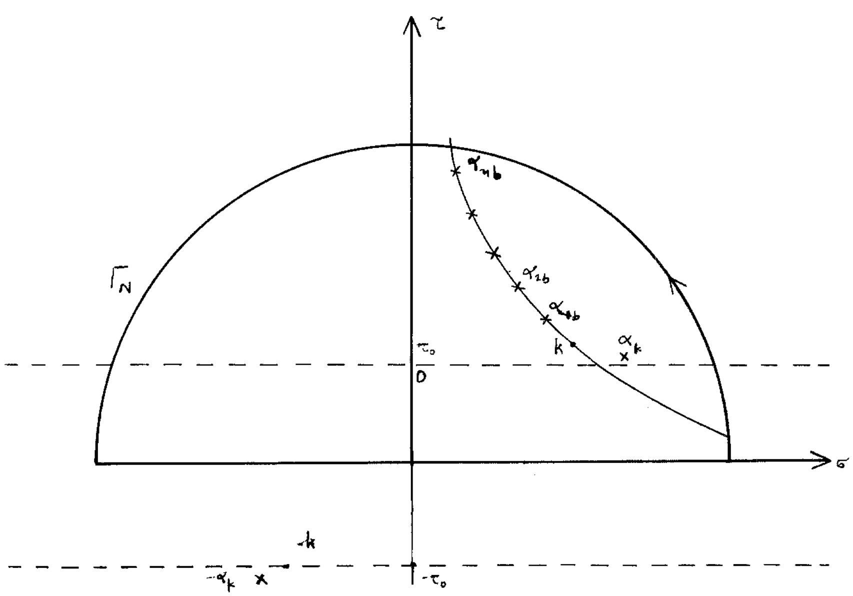

Fig. 2. The upper half α-plane and the poles of $\Phi(\alpha, y)$ on it.

only singularities of $\Phi(\alpha, y)$ in $\tau>0$ are the poles at

$$\alpha=\alpha_k=k\cosh\beta \quad \text{and} \quad \alpha=\alpha_{nb}=i\left(n^2\pi^2/b^2-k^2\right)^{1/2}, \qquad n=1,2,\ldots$$

It can be shown that if the contour is closed by a sequence of concentric circular arcs Γ_N, $N=1,2,\ldots$, such that each arc Γ_N, whose equation is $|\alpha|=|R_N|$, passes through no pole of $\Phi(\alpha, y)$ and the radius R_n tends to infinity as $N\to\infty$, then $\Phi(\alpha, y)\to 0$ on Γ_N as $N\to\infty$. Therefore

$$\int_\Gamma \Phi(\alpha, y)e^{-i\alpha x}d\alpha=0$$

on Γ where Γ is the limit of Γ_N, $N\to\infty$ and thus from the residue theorem we have

$$\phi(x, y)=i\Sigma\,\mathrm{Res}\left\{\Phi(\alpha, y)e^{-i\alpha x}\right\}$$

The calculation of the residues at the above poles gives

$$\begin{aligned}\phi(x, y)= &-\frac{k\sin\beta M_+^2(k\cosh\beta)\exp(-kb\sinh\beta)}{\sinh(kb\sinh\beta)}\\ &\times\exp\left[k(-ix\cosh\beta+y\sinh\beta)\right]\\ &+\frac{2i\pi E}{b^2}\sum_{n=1}^{\infty}\frac{n(-)^n M_+(\alpha_n)}{\alpha_{nb}\left[(\pi n^2/b)+(\alpha_{nb}\tanh\beta)^2\right]}\\ &\times\left(\frac{n\pi}{b}\cos\frac{n\pi}{b}y+\alpha_{nb}\tanh\beta\sin\frac{\pi n}{b}y\right)e^{-i\alpha_{nb}x} \qquad (5.3)\end{aligned}$$

The first part of the right-hand side of (5.3) represents the reflected Kelvin wave traveling in the duct ($0\leqslant y\leqslant b$, $x<0$) to the left.

The remaining part of the solution expresses the set of modes existing along the above duct. For $bk<\pi$ all α_{nb}'s become positive imaginary and all these modes represent attenuated stationary waves whose amplitudes decrease exponentially as x decreases. For $x>0$ the contour may be closed in the lower half-plane in a manner similar to the above (Figure 3). It can be shown again that $\Phi(\alpha, y)\to 0$ on the lower semicircular arc, as the radius of this arc tends to infinity, and thus

$$\phi(x, y)=-i\Sigma\,\mathrm{Res}\left\{\Phi(\alpha, y)e^{-i\alpha x}\right\}$$

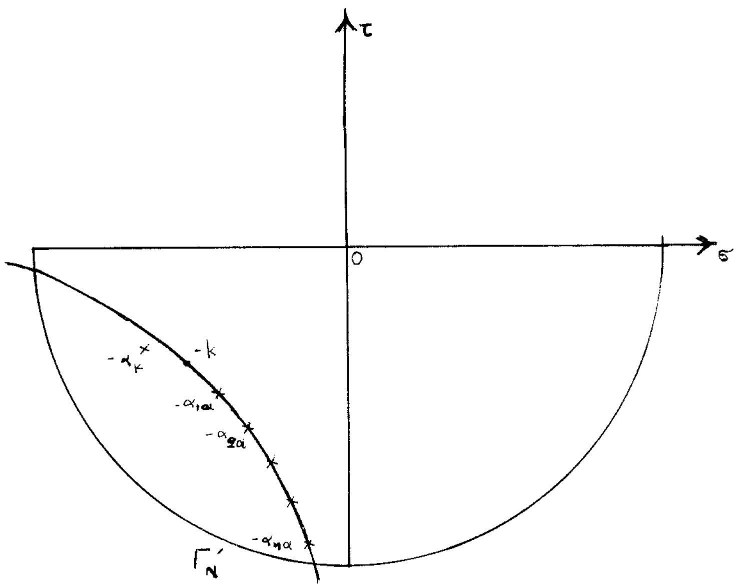

Fig. 3. The lower half α-plane and the poles of $\Phi(\alpha, y)$ on it.

since the singularities of $\Phi(\alpha, y)$ in $\tau<0$ are only poles at

$$\alpha = -\alpha_k = -k\cosh\beta \quad \text{and} \quad \alpha = -\alpha_{na} = -i\left(\frac{\pi^2 n^2}{a^2} - k^2\right)^{1/2}$$

as it can be easily seen from (5.2) if we replace $M_+(\alpha)$ by $M(\alpha)/M_-(\alpha)$.
After the calculation of the residues at the above poles we find

$$\begin{aligned}\phi(x, y) = &-\frac{\sinh[k(a-b)\sinh\beta]\exp(-kb\sinh\beta)}{\sinh(ka\sinh\beta)} \\ &\times \exp[k(ix\cosh\beta - y\sinh\beta)] \\ &-\frac{2iE}{\alpha}\sum_{n=1}^{\infty}\frac{\sin(\pi nb/2)}{\alpha_{na}M(\alpha_{na})\left[(\pi n/a)^2 + (\alpha_{na}\tanh\beta)^2\right]} \\ &\times\left(\frac{\pi n}{a}\cos\frac{\pi n}{a}y - \alpha_{na}\tanh\beta\sin\frac{\pi n}{a}y\right)e^{i\alpha_{na}x}\end{aligned} \tag{5.4}$$

The first part of (5.4) represents a Kelvin wave traveling in the direction of x increasing.

The second part (5.4) is the set of the existing modes in this part of the field. For $ka<\pi$ all these modes are attenuated standing waves whose amplitude decays exponentially to zero in the positive x direction.

5.2. Region B ($b \leqslant y \leqslant a, -\infty < x < \infty$). From equations (3.3) and (4.9) we get

$$\Phi(\alpha, y) = -E\frac{M_+(\alpha)}{\sinh\gamma(a-b)}\left[\frac{e^{\gamma(a-y)}}{\gamma+\alpha\tanh\beta}+\frac{e^{-\gamma(a-y)}}{\gamma-\alpha\tanh\beta}\right] \tag{5.5}$$

Obviously $\Phi(\alpha, y)$ has no branch points and the only singularities in it are (simple) poles coming from the poles of $M_+(\alpha)$, the zeros of $\sinh\gamma(a-b)$ (except $\gamma=0$) and the zeros of $(\gamma\pm\alpha\tanh\beta)$.

To determine $\phi(x, y)$ for $x<0$ we follow the same procedure as in region A ($x<0$). Then we find

$$\begin{aligned}\phi(x, y) = {} & \frac{k\sinh\beta\, M_+^2(k\cosh\beta)\exp[-k(a+b)\sinh\beta]}{\sinh(kd\sinh\beta)} \\ & \times\exp[k(-ix\cosh\beta+y\sinh\beta)] \\ & -\frac{2i\pi E}{d^2}\sum_{n=1}^{\infty}\frac{n(-)^n M_+(\alpha_{nd})}{(\pi n/2)^2+(\alpha_{nd}\tan\beta)^2}\left\{\frac{\pi n}{d}\cos\left[\frac{\pi n}{d}(y-a)\right]\right. \\ & \left.+\alpha_{nd}\tanh\beta\sin\left[\frac{\pi n}{d}(y-a)\right]\right\}e^{-i\alpha_{nd}x}\end{aligned} \tag{5.6}$$

where $d=a-b$.

The Kelvin wave in (5.6) propagates to the left in the "geometric shadow" zone and its amplitude is

$$\exp(kb\sinh\beta)\cdot\frac{\sinh(kb\sinh\beta)}{\sinh(kd\sinh\beta)}$$

times less than the amplitude of the reflected Kelvin wave in the semi-infinite channel ($0\leqslant y\leqslant b, x<0$) [cf. equation (5.3)]. The set of the modes in the series of (5.6) may also involve traveling waves in the shadow zone with constant amplitude. This happens if $kd>\pi$. For $kd<\pi$ all the modes present standing attenuated waves.

Lastly, for $x>0$ the solution is given, apart from an incident wave ϕ_i [see equation (2.2)], by (5.4) as (it is expected and) can be easily verified if we take the inverse Fourier transform of (5.5) following the same lines as in the case of region A ($x>0$).

6. THE CASE OF ELIMINATION OF THE PLATE $y=\alpha$

Assuming that the boundary $y=a$ is removed to infinity, i.e., $a\to\infty$ with b finite, we are led to the problem of diffraction by an infinite and a semi-infinite barrier discussed in a previous paper (Kapoulitsas, 1979) and we might expect that the solution of the present closed-field problem should be extended to include the solution of the previous open-field problem in the limit $\alpha\to\infty$.

Upon this we note the following:

(i) In section $x<0$, $0\leqslant y\leqslant b$ of the field the solution (5.3) of the present problem differs from the solution of the previous problem only as regards the term $M+(\alpha)$, which is replaced there by $b^{1/2}L+(\alpha, b)$ [cf. equation (5.5) in the above paper], where

$$L_+(\alpha, l)=L_-(-\alpha, l)=\left(\frac{\sin kl}{kl}\right)^{1/2}\exp\left\{\frac{il\alpha}{\pi}\left[1-C+\log\left(\frac{2\pi}{kl}\right)+\frac{i\pi}{2}\right]\right\}$$
$$\times\left\{\exp\left[\frac{il\gamma}{\pi}\log\left(\frac{\alpha-\gamma}{k}\right)\right]\right\}\prod_{n=1}^{\infty}\left(1+\frac{\alpha}{\alpha_{nl}}\right)e^{i\alpha l/n\pi} \tag{6.1}$$

(C is Euler's constant) and

$$L(\alpha, l)=L_+(\alpha, l)\cdot L_-(\alpha, l)=\frac{\sinh l\gamma}{l\gamma e^{l\gamma}} \tag{6.2}$$

(cf. Mittra and Lee, 1971, p. 113). First we observe from (4.1) as $\alpha\to\infty$, $|\mathrm{Im}\,\alpha|<k_2$,

$$\lim_{a\to\infty} M(\alpha)=\lim_{a\to\infty}\frac{\sinh\gamma b}{\gamma}\cdot\frac{e^{(a-b)\gamma}-e^{-(a-b)\gamma}}{e^{\gamma a}-e^{-\gamma a}}=b\frac{\sin\gamma b}{\gamma b e^{\gamma b}}=bL(\alpha, b).$$

To identify the solution of (5.3) for $\alpha\to\infty$ with that in the open-field problem it remains to prove that $\lim M_+(\alpha)=b^{1/2}L_+(\alpha)$.

Yet, $M_+(\alpha)$ can be written as

$$M_+(\alpha)=L_+(\alpha, b)\cdot\frac{d^{1/2}L_+(\alpha, d)}{a^{1/2}L_+(\alpha, a)}b^{1/2}$$

as it is found from (4.3) and (6.1). Therefore

$$\lim_{a\to\infty} M_+(\alpha)=L_+(\alpha, b)\lim_{a\to\infty}\frac{d^{1/2}L_+(\alpha, d)}{a^{1/2}L_+(\alpha, a)}b^{1/2}=b^{1/2}L_+(\alpha, b)$$

since, as can be found, $\lim_{l\to\infty}l^{1/2}L_+(\alpha, l)$ exists and is finite.

(ii) In section $x>0$ the first part of the solution (5.4) (Kelvin wave) becomes

$$\exp(-2kb\sinh\beta)\varphi_i$$

On the other hand each term of the series of (5.4) tends to zero as $a\to\infty$, while the sum of the series is indeterminate.

So the solution regarding finite a cannot be extended to describe the field in the region $x>0$ for $a\to\infty$.

We might note that the above sum corresponds, as $a\to\infty$, to the integral coming from the path along the branch-point cut in the correspondent open-field problem.

A similar situation appears in the region $x<0, b\leqslant y\leqslant a$ as follows from the solution (5.6).

7. DISCUSSION

We have just seen that in all sections of the field there exists an appropriate secondary unattenuated wave (the Kelvin wave) traveling with a constant velocity $c=(gh)^{1/2}$.

The amplitude of this Kelvin wave, as a function of k, β, and l ($l=a$, b, or d), depends on the breadth (l) of the relevant section of the field and the depth (h) of the water, as well as on the frequency (ω) of the incident wave and the angular velocity (f) of the rotation of the system. The appearance of Kelvin waves, which are the only nonharmonic in y waves in the field of the rotating channel, is perhaps the most obvious consequence of the effect of rotation.

It must also be noted that for $k=\pi n/l$ ($l=a,b,d$) the amplitude of the corresponding nth term of the series in (5.3), (5.4), and (5.6), respectively, becomes infinitely large as then $\alpha_{nl}=0$ (resonance). Thus, we must suppose that k is sufficiently far from $\frac{\pi n}{l}$, $n=1,2,\ldots$

The remaining part of the solution, in all sections of the field, is a convergent series containing all possible modes which, for $ka<\pi$, are all standing attenuated waves in the direction of their propagation. For $f=0$ the solution concerns the surface elevation of the water without rotation, and also represents the solution of the corresponding problems in acoustics and electromagnetism, where the exitation is due to a plane harmonic wave.

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